Recursion Formula for the Probability Distribution of Sum of (v-u+1)-sided die and some of its Properties

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1. INTRODUCTION
Recursion Formula is any formula (equation) involving two or more terms of a sequence. Specific examples of such formulas include the Fibonacci formula, Lucas formula, etc. Now let $X$ be a random variable associated with a random experiment on a nonempty set $E$ with two possible outcomes, then $X$ is said to have a Bernoulli distribution $B(p)$ given by

$$b(x; p) = p^x(1 - p)^{1-x}; x = 0, 1 \quad (1.1)$$

Where the parameter $p$ is the probability of success.

If we extend the range (domain) of the independent variable $x$ to $\{0,1,2,\ldots,n\}$ we have the Binomial distribution $B(n,p)$ given by

$$b(x; n, p) = \binom{n}{x} p^x (1 - p)^{n-x}; x = 0, 1, 2, \ldots, n \quad (1.2)$$

Where the parameter $n$ is the number of independent trials.

Suppose we impose a further condition on the domain of $X$, that is selection (sampling) is without replacement, then the number of trials (no longer independent) gives rise to Hypergeometric distribution $H(k,n,N)$ given by

$$h(x; k,n,N) = \binom{k}{x} \binom{n-k}{n-x} \quad L \leq x \leq U \quad (1.3)$$

Where $k,n,N$ are fixed constants, $L = max\{0, k - N + n\}$ and $U = min\{n, k\}$.

Now, suppose we decide to fix the number of successes we require in (1.2) and then observe the random number of trials needed to obtain this number of successes, then the random number $X$ of trials required to obtain the first success has a geometric distribution given by

$$g(x; p) = pq^{x-1}; x = 1, 2, 3, \ldots \quad (1.4a)$$
and if the random variable $X$ is the number of failures before the occurrence of the first success, then we have

$$g(x; p) = pq^x; x = 0, 1, 2, 3, \ldots \quad (1.4b)$$

Observe that the geometric distributions in (1.4a) and (1.4b) are distributions of the number of independent Bernoulli trials required to obtain a single success. Hence, a further generalization is to seek for the distribution of the random variable $X$ on which the $r$th success $r > 1$ occurs, such a distribution is called the negative binomial distribution $NB(r, p)$ and is given by

$$nb(x; r, p) = \binom{x - 1}{r - 1} p^r q^{x-r}; x = r, r + 1, r + 2, \ldots \quad (1.5a)$$

and if the random variable $X$ is the number of failures before the occurrence of the first $r$th success, then we have

$$nb(x; r, p) = \binom{x + r - 1}{r - 1} p^r q^x; x = 0, 1, 2, \ldots \quad (1.5b)$$

One of the most important generalizations of (1.2) above is the discrete multivariate distribution function that belong to the (one dimensional) multinomial distribution $M(n, p_1, \ldots, p_k)$ which is given by

$$m(x_1, x_2, \ldots, x_k) = \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}; \quad (1.6)$$

Where $\sum_{i=1}^{k} p_i = 1$ and $n, p_1, p_2, \ldots, p_k$ are the parameters.

To mention but a few, the probability mass functions considered in (1.1) to (1.6) are often referred as the classical or standard discrete probability mass functions. One major weakness of most of these standard pmf is its inadequacy in modeling different types of data set.

Consequent to the above weakness of standard pmf; in modeling different types of data set, in recent times, researchers have focused more on generalizing-improving probability distribution functions with the aim of making the functions to be more robust, accommodating and applicable in modeling different types of data. In order to improve on the discrete models (1.1) to (1.6) we consider some of the important contributors and their results in the sequel.

According to Philippou and Muwafi (1982), Philippou et al. (1983), they introduced the distribution of order $k$ which gives rise to several studies of distribution of order $k$ as contained in the reference (which reduce to the respective classical probability distribution for $k = 1$) some of these distributions are given by

$$b(x; k, n, p) = \sum_{j=1}^{k-1} \sum_{x_1, x_2, \ldots, x_k} \binom{x_1 + x_2 + \cdots + x_k, x_1, x_2, \ldots, x_k}{x, x_1, x_2, \ldots, x_k} p^x \left( \frac{q}{p} \right)^{\sum_{i=1}^{k} x_i}; x = 0, 1, \ldots, \left[ \frac{n}{k} \right] \quad (1.7)$$

Where $x_1 + 2x_2 + \cdots + kx_k = n - kx - j$, $[a]$ is the greatest integer function less than or equal to $a$

$$g(x; k, p) = \sum_{x_1, x_2, \ldots, x_k} \binom{x_1 + x_2 + \cdots + x_k, x_1, x_2, \ldots, x_k}{x, x_1, x_2, \ldots, x_k} p^x \left( \frac{q}{p} \right)^{\sum_{i=1}^{k} x_i}; x \geq k \quad (1.8)$$

Where $x_1 + 2x_2 + \cdots + kx_k = x - k$.

$$nb(x; k, p) = \sum_{x_1, x_2, \ldots, x_k} \binom{x_1 + x_2 + \cdots + x_k, r - 1, x_1, x_2, \ldots, x_k, r - 1}{x, x_1, x_2, \ldots, x_k, r - 1} p^x \left( \frac{q}{p} \right)^{\sum_{i=1}^{k} x_i}; x \geq rk \quad (1.9)$$

Where $x_1 + 2x_2 + \cdots + kx_k = x - rk$.

represents the binomial, geometric, negative binomial distribution of order $k$ respectively. The symptomatic properties of some of these distributions give rise to other important distributions as studied by Aki et al. (1984), Feller (1968).

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Panaretos and Xekalaki (1986b) improved on some of the above distributions, in particular (1.2) and (1.3) via sampling from an urn containing \(a\) white balls and \(b\) black balls. The following **Hypergeometric distribution of order \(k\)** was introduced. Assuming that \(n\) balls are drawn one at a time;

Without replacement gives;

\[
h_1(x; k, n, p) = \sum_{j=1}^{k-1} \sum_{x_1, x_2, \ldots, x_k} \left( \frac{x_1 + x_2 + \cdots + x_k}{x_1, x_2, \ldots, x_k} \right) b^{x_1} \frac{a^{n-x_1}}{(a+b)^n} \; ;
\]

\[x = 0, 1, \ldots, \left[\frac{n}{k}\right]\] (1.10a)

With replacement gives

\[
h_2(x; k, n, p) = \sum_{j=1}^{k-1} \sum_{x_1, x_2, \ldots, x_k} \left( \frac{x_1 + x_2 + \cdots + x_k}{x_1, x_2, \ldots, x_k} \right) (\frac{a}{a+b})^{n-x_1} (\frac{b}{a+b})^{x_1} \; ;
\]

\[x = 0, 1, \ldots, \left[\frac{n}{k}\right]\] (1.10b)

With replacement and addition of one ball of the same colour that was selected, before the next draw gives

\[
h_3(x; k, n, p) = \sum_{j=1}^{k-1} \sum_{x_1, x_2, \ldots, x_k} \left( \frac{x_1 + x_2 + \cdots + x_k}{x_1, x_2, \ldots, x_k} \right) \frac{b^{x_1}}{c^{n-x_1}} \frac{a^{n-x_1}}{(a+b)^n} \; ;
\]

\[x = 0, 1, \ldots, \left[\frac{n}{k}\right]\] (1.10c)

With replacement and addition of \(c\) balls of the same colour that was selected, before the next draw gives

\[
h_4(x; k, n, p) = \sum_{j=1}^{k-1} \sum_{x_1, x_2, \ldots, x_k} \left( \frac{x_1 + x_2 + \cdots + x_k}{x_1, x_2, \ldots, x_k} \right) \frac{b^{x_1}}{c^{x_1}} \frac{a^{n-x_1}}{(a+b)^n} \; ;
\]

\[x = 0, 1, \ldots, \left[\frac{n}{k}\right]\] (1.10d)

Where \(x_1 + x_2 + \cdots + x_k = n - kx - j\), \(a^{(m)} = a(a-1) \cdots (a-m+1)\)

\(a_{(m)} = a(a+1) \cdots (a+m-1)\)

Panaretos and Xekalaki (1986c) introduced the **Cluster Binomial Distribution** as an improvement on the classical binomial distribution via sampling from an urn containing \(i\) labeled balls \((i = 1, 2, \ldots, k)\) with \(p_i\) probability that a ball bearing the number \(i\) will be drawn such that \(\sum_{i=1}^{k} p_i = p\). Then, \(q = 1 - p\) is the probability that a ball bearing a zero will be drawn. Let \(X\) be a random variable that count the sum of the numbers on the balls drawn. If the random variable \(X\) take the value \(r\) for the \(n\) balls drawn, \(r_1\) bear the number 1, \(r_2\) bear the number 2 and so on, \(r_k\) bear the number \(k\) so that \(\sum_{i=1}^{k} r_i = r\) and each of the remaining \(n - \sum_{i=1}^{k} r_i\) balls bear the zero. Then the pmf is given by;

\[
\text{cb}(r; n, k, p_1, \ldots, p_k) = \sum_{r_1, r_2, \ldots, r_k} \left( \begin{array}{c} n \\ r_1, r_2, \ldots, r_k, n - \sum_{i=1}^{k} r_i \end{array} \right) \left( \prod_{i=1}^{k} p_i^{r_i} \right) q^{n-\sum_{i=1}^{k} r_i} \; (1.11)
\]

Abraham De Moivre (1756), (republished in 1967), studied the probability distribution for a fair (balanced) \(m\)-sided die tossed \(n\) number of times. Let \(X_{m}^{(n)}\) be a random variable that count the total score in \(n\) rolls of an \(m\)-sided die, the following probability mass function was obtained

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\[ P(X_n^{(m)} = x) = \frac{\beta_1}{m^n} \sum_{s=0}^{\beta_1} (-1)^s \binom{n}{s} \binom{n-1+x-ms}{n-1}; 0 \leq x \leq (m-1)n \]  

(1.12)

Where \( \beta_1 = \min\{n, \lfloor \frac{x}{m} \rfloor \} \) and \( \lfloor \frac{x}{m} \rfloor \) is the greatest integer function less than or equal to \( \frac{x}{m} \).

The coefficient of \( \frac{1}{m^n} \) often denoted by \( C_m(n, x) \) have been studied in detail by Dafnis (2007), Freund (1956), who discussed their role in occupancy theory. In particular, \( C_m(n, x) \) can be interpreted as "the number of ways of putting \( n \) indistinguishable objects into \( x \) numbered boxes with each box containing at most \( m-1 \) objects. So that if \( m = 2 \) we have the standard binomial coefficient given by

\[ C_2(n, x) = \binom{n}{x}; 0 \leq x \leq n \]  

(1.13)

A recurrence formula for computing \( C_m(n, x) \) is given by

\[ C_m(n, x) = \sum_{j=0}^{m-1} C_m(n-1, x-j) \]  

(1.14)

One can easily see that for \( m = 2 \), this recursion reduces to the well-known classical binomial identity.

The number \( C_m(n, x) \) has been used extensively in probability studies (De Moivre (1756), Feller (1968), Balakrishnan et al. (1993), Balasubramanian (1995), Makri and Philippou (2005, 2007a, 2007b) and related areas like reliability and inferential statistics Gabai (1970), Bollinger and Burchard (1990), Ailing (1993). For more properties on \( C_m(n, x) \): generalized Pascal triangles or Pascal triangles of order \( m \), we refer to Fraund (1956), Gabai (1970), Bondarenko (1993), Ollerton and Shannon (1998, 2004, 2005), Dafnis (2007) and the references therein.

Balasubramanian et al. (1995) introduced the extended binomial distribution of order \( m \) with index \( n \) and parameter \( p \) as an improved version of the standard binomial distribution and the distribution studied by Abraham De Moivre in (1756) via considering \( n \) roll of an \( m \) sided die which is not necessarily fair (balanced) with face marked \( i \) \( (i = 0, 1, 2, \ldots, m - 1) \) and a turn-up side probability \( p_i \) \( (\sum_{i=0}^{m-1} p_i = 1) \) satisfying the condition \( q^m - p^m = q - p \). It was proved that if \( X_n^{(m)} \) is a random variable that count the total score in \( n \) rolls of an \( m \)-sided die then the probability mass function (pmf) is given by

\[ P(X_n^{(m)} = x; p) = \sum_{s=0}^{\beta_1} (-1)^s \binom{n}{s} \binom{n-1+x-ms}{n-1} p^s q^{(m-1)n-x}; 0 \leq x \leq (m-1)n \]  

(1.15)

Where \( \beta_1 = \min\{n, \lfloor \frac{x}{m} \rfloor \} \) and \( \lfloor \frac{x}{m} \rfloor \) is the greatest integer function less than or equal to \( \frac{x}{m} \).

Observe that if the die is a fair one, then it implies that \( p = \left( \frac{1}{m} \right)^{m-1} = q \) so that on substitution into equation (1.15) yield the result of Abraham De Moivre.

Ashok et al (2011) studied and derived a recursion formula for the probability distribution of the sum of rolling a fair dice (6-sided die) \( n \) times (which is equivalently to rolling \( n \) several dice once) which is given by:

\[ f_j(m) = \frac{1}{6} \left( f_{j-1}(m-1) + f_{j-1}(m-2) + \cdots + f_{j-1}(m-6) \right); j = 1, 2, \ldots, n; m \in [j, 6j] \]  

(1.16)

Okoli (2017) studied a \((v-u+1)\)-sided die with turn-up side probability denoted by \( T(x, y) \) such that \( T(x, y) = p^x q^y \): \( x, y = u, u+1, u+2, u+3, \ldots, v \); \( x + y = k; 0 \leq p, q \leq 1 \). It was showed that the generating function \( G(t) \) for the \((v-u+1)\)-sided die is given by

\[ G(t) = q^{k-v} p^u t^u \frac{(q^{v-u+1} - p^{v-u+1} t^{v-u+1})}{q - pt} \]  

(1.17)
Satisfying the normalization condition
\[ p^u(q^{v-u+1} - p^{v-u+1}) = q^{v-k}(q-p) \quad (1.18) \]
The following theorem was proved.

**Theorem 1.1**

Let \( X_n^{(v-u+1,k-u+1)} \) be a random variable that count the total score in \( n \) rolls of a \((v-u+1)\)-sided die and turn-up side probabilities \( T(x, k-x) \) satisfying the condition \( p^u(q^{v-u+1} - p^{v-u+1}) = (q-p) \), with range \( x = u, u+1, u+2, u+3, ..., v \). Then the probability mass function (pmf) is given by
\[
P(X_n^{(v-u+1,k-u+1)} = x; p) = \frac{\beta_2}{\binom{n}{s}} \sum_{s=0}^{x} (-1)^s \binom{n}{s} \left( \frac{n-1+x-(v-u+1)s-un}{n-1} \right) p^s q^{kn-x};
\]
where \( \beta_2 = \min \left\{ n \left\lfloor \frac{x-un}{v-u+1} \right\rfloor \right\}, k \leq v. \)

Also the following corollary for a fair (balanced) die (i.e. \( k = v \), \( q = \left( \frac{1}{v-u+1} \right)^2 \)) was obtained

**Corollary 1.2**

Let \( X_n^{(v-u+1,p-v+1)} \) be a random variable that count the total score in \( n \) rolls of a \((v-u+1)\)-sided die and turn-up side probabilities \( T(x, v-x) \) satisfying the condition \( p^u(q^{v-u+1} - p^{v-u+1}) = (q-p) \), with range \( x = u, u+1, u+2, u+3, ..., v \). Then the probability mass function (pmf) is given by
\[
P(X_n^{(v-u+1,p-v+1)} = x; p) = \frac{\beta_3}{\binom{n}{s}} \sum_{s=0}^{x} (-1)^s \binom{n}{s} \left( \frac{n-1+x-(v-u+1)s-un}{n-1} \right) p^s q^{vn-x};
\]
where \( \beta_3 = \min \left\{ n \left\lfloor \frac{x-un}{v-u+1} \right\rfloor \right\}, u \leq x \leq vn. \)

**Corollary 1.3**

Let \( X_n^{(v-u+1,p-v+1)} \) be a random variable that count the total score in \( n \) rolls of a \((v-u+1)\)-sided fair die and turn-up side probabilities \( T(x, v-x) \) satisfying the condition \( p^u(q^{v-u+1} - p^{v-u+1}) = (q-p) \), with range \( x = u, u+1, u+2, u+3, ..., v \). Then the probability mass function (pmf) is given by
\[
P(X_n^{(v-u+1,p-v+1)} = x; p) = \frac{1}{(v-u+1)^n} \frac{\beta_3}{\binom{n}{s}} \sum_{s=0}^{x} (-1)^s \binom{n}{s} \left( \frac{n-1+x-(v-u+1)s-un}{n-1} \right) p^s q^{vn-x};
\]
where \( u \leq x \leq vn. \)

Motivated by the results of the research in this area, we seek to derive a recursion formula for the probability distribution of an arbitrary (not necessarily fair) \((v-u+1)\)-sided die with turn-up side probabilities \( T(x, v-x) \) satisfying the condition \( p^u(q^{v-u+1} - p^{v-u+1}) = (q-p) \), with range \( x = u, u+1, u+2, u+3, ..., v. \)

**2. METHODOLOGY**

2.1 A Recursion for Distribution of the Sum of rolling \( n \) of \((v-u+1)\)-sided Die.

Let \( k \in \mathbb{N} \) (where \( v \) is not necessarily equal to \( k \)) and if we defined the turn-up side probabilities as
\[
T(x, y) = p^x q^y: x, y = u, u+1, u+2, u+3, ..., v; x + y = k; 0 \leq a \leq p, q \leq 1 = b. \quad (2.1)
\]
Where \( u < k \leq v \)
Then equation (2.1) describes a \((v - u + 1)\)-sided die with turn-up side probability \(T(x, y)\), so that if
\[
\begin{align*}
(a) & \quad u = 0 \text{ and } k = v = m - 1; \implies v - u + 1 = m; \text{ which is the } m\text{-sided die studied by Balasubramanian et al. (1995).} \\
(b) & \quad u = 0 \text{ and } k \leq v = m - 1; \implies v - u + 1 = m; \text{ which is the } m\text{-sided die studied by Okoli (2017).} \\
(c) & \quad u = 1 \text{ and } k \leq v = m; \implies v - u + 1 = m; \text{ which is the } m\text{-sided die studied by Okoli (2017).}
\end{align*}
\]
However, there is no loss of generality if we assume that \( k = v \). Now, let \( x_j \in \{u, u + 1, \cdots, v\} \) be the number that turns up when the \( j \)th die is rolled for each \((v - u + 1)\)-sided die for \( j = 1, 2, \cdots, n \). It then follows that the probability distribution for each \( x_j \) is given by
\[
f(x) = p^x q^{v-x}; x = u, u + 1, \cdots, v \quad (2.2)
\]
Equation (2.2) implies that \( P\{x_j = x\} = f_j(x) = f(x) \) for each \((v - u + 1)\)-sided die, so that if we define the random variable \( S_j^{(v-u+1)} \) to be the sum of the \( j \) rolled of each \((v - u + 1)\)-sided die such that \( P\{S_j^{(v-u+1)} = x\} = f_j(x) \) and \( \sum_{x=u}^v f_j(x) = 1 \).

Then at this juncture, we shall now proceed to introduce the results of this work in the theorems that follows in the next section.

3. MAIN RESULTS

Theorem 3.1

Let \( S_j^{(v-u+1)} \) be a random variable that count the total score in \( j \) rolled of a \((v - u + 1)\)-sided die and turn-up side probabilities \( T(x, v - x) \) satisfying the condition \( p^n(q^{v-u+1} - p^{v-u+1}) = (q - p) \), with range \( x = u, u + 1, u + 2, u + 3, \ldots, v \). Then the recursion formula for the probability mass function (pmf) is given by
\[
P\{S_j^{(v-u+1)} = r\} = \sum_{x=u}^v f_{j-1}(r-x) p^x q^{v-x}; \quad ju \leq r \leq jv, j = 1, 2, \cdots, n
\]

Proof

Let \( r \in [ju, jv] \), then for any event \( S_j^{(v-u+1)} = r \) we have that
\[
\{S_j^{(v-u+1)} = r\} = \bigcup_{x=u}^v \{S_{j-1}^{(v-u+1)} = r-x, x_j = x\}
\]
Which implies that
\[
P\{S_j^{(v-u+1)} = r\} = P\left(\bigcup_{x=u}^v \{S_{j-1}^{(v-u+1)} = r-x, x_j = x\}\right)
\]
\[
= \sum_{x=u}^v P\{S_{j-1}^{(v-u+1)} = r-x, x_j = x\} = \sum_{x=u}^v P\{S_{j-1}^{(v-u+1)} = r-x\} P\{x_j = x\}
\]
\[
= \sum_{x=u}^v f_{j-1}(r-x)f(x) = \sum_{x=u}^v f_{j-1}(r-x) p^x q^{v-x}
\]
If we are dealing with a fair (balanced) die \( i.e \ q = \left(\frac{1}{v-u+1}\right)^2 = p \) then the corollary that follows is a consequence of theorem 3.1 above.
Corollary 3.2

Let $S_j^{(v-u+1)}$ be a random variable that count the total score in $j$ rolled of a $(v - u + 1)$-sided fair die and turn-up side probabilities $T(x, v - x)$ satisfying the condition $p^u(q^{v-u+1} - p^{v-u+1}) = (q - p)$, with range $x = u, u + 1, u + 2, u + 3, \ldots, v$. Then the recursion formula for the probability mass function (pmf) is given by

$$P \left( \left\{ S_{j}^{(v-u+1)} = r \right\} \right) = \frac{1}{(v - u + 1)} \sum_{x=u}^{v} f_{j-1}(r - x); \ j u \leq r \leq jv, j = 1, 2, \ldots, n. $$

Also, to obtain recurrence formula that conform to the probability distribution due to Balasubramanian et al. (1994) and Okoli (2017) for the case of a fair die, we simply put $u = 0, v = m - 1$ and $u = 1, v = m$ to obtain the corollaries that follows.

Corollary 3.3

Let $S_j^{(m)}$ be a random variable that count the total score in $j$ rolled of a $(m)$-sided fair die and turn-up side probabilities $T(x, m - 1 - x)$ satisfying the condition $(q^m - p^m) = (q - p)$, with range $x = 0, 1, 2, \ldots, m - 1$. Then the recursion formula for the probability mass function (pmf) is given by

$$P \left( \left\{ S_{j}^{(m)} = r \right\} \right) = \frac{1}{m} \sum_{x=0}^{m-1} f_{j-1}(r - x); \ 0 \leq r \leq j(m - 1), j = 1, 2, \ldots, n.$$ 

Now, we shall give some implications of corollary 3.3 in relation to recurrence probability distribution formula which yield the classical Bernoulli’s distributions for a coin tossed once. Observe that if $m = 2$ and $n = 1$ then we have that the recursion formula reduces to the Bernoulli’s distribution given by

$$P \left( \left\{ S_1^{(2)} = r \right\} \right) = \frac{1}{2} (f_0(r) + f_0(r - 1)) = \frac{1}{2}; \ r = 0, 1$$

Corollary 3.4

Let $S_j^{(m)}$ be a random variable that count the total score in $j$ rolled of a $(m)$-sided fair die and turn-up side probabilities $T(x, m - x)$ satisfying the condition $p^m(q^m - p^m) = (q - p)$, with range $x = 1, 2, 3, \ldots, m$. Then the recursion formula for the probability mass function (pmf) is given by

$$P \left( \left\{ S_{j}^{(m)} = r \right\} \right) = \frac{1}{m} \sum_{x=1}^{m} f_{j-1}(r - x); \ j \leq r \leq jm, j = 1, 2, \ldots, n.$$ 

Observe that in corollary 3.4, if we put $m = 6$, we obtain the corollary that follows.

Corollary 3.5

Let $S_j^{(6)}$ be a random variable that count the total score in $j$ rolled of a fair dice and turn-up side probabilities $T(x, 6 - x)$ with range $x = 1, 2, \cdots, 6$. Then the recursion formula for the probability mass function (pmf) is given by

$$P \left( \left\{ S_{j}^{(6)} = r \right\} \right) = \frac{1}{6} \sum_{x=1}^{6} f_{j-1}(r - x); \ j \leq r \leq 6j, j = 1, 2, \cdots, n.$$ 

It is important to note that corollary 3.5 is actually the result obtained by Ashok et al. (2011) as specified in equation (1.16) of section one.

Theorem 3.6

Let $S_j^{(v-u+1)}$ be a random variable that count the total score in $j$ rolled of a $(v - u + 1)$-sided fair die and turn-up side probabilities $T(x, v - x)$ satisfying the condition $p^u(q^{v-u+1} - p^{v-u+1}) = (q - p)$, with range $x = u, u + 1, u + 2, u + 3, \ldots, v$. Then the moment generating function (mgf) is given by
\[ G_n(t) = E \left( e^{t S_j^{(p-u+1)}} \right) = \left[ p^n \frac{q^{v-u+1} - p^{v-u+1} e^{t(v-u+1)}}{q - pe^t} \right]^n \]

**Proof**

Now, observe that the moment generating function (mgf) of \( S_j^{(p-u+1)} \) constitute a convolution of \( X_j^{(p-u+1)} (j = 1, 2, 3, \ldots n) \). Where each \( X_j^{(p-u+1)} \) is an independent identically distributed (iid) random variables corresponding to the scores of \((v - u + 1)-sided die and turn-up side probabilities \( T(x, v - x) \). Thus

\[ G_n(t) = E \left( e^{t S_j^{(p-u+1)}} \right) = E \left( e^{t \sum_{j=1}^{n} X_j^{(p-u+1)}} \right) = \prod_{j=1}^{n} E \left( e^{t X_j^{(p-u+1)}} \right) = \left[ p^n \frac{q^{v-u+1} - p^{v-u+1} e^{t(v-u+1)}}{q - pe^t} \right]^n \]

This completes the proof.

The results obtained in this research work unifies and improves the works of previous researchers in this direction, haven shown that the existing results in the literature can be deduce easily from the results in this paper.

**REFERENCES**


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