Modified Geometric Distribution of Certain Order and Some of its Properties

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ABSTRACT

In this research work, we present a new distribution function that modifies and generalize the geometric distribution of order k which was introduced by Philippou *et al.* (1983). The associated statistics, generating function, and its relationship with some other discrete distributions are established. The results obtained in this paper improve, generalize and complement the works of several authors in the literature.

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1. Introduction and Preliminaries

Philippou *et al.* (1983) introduced the distribution of the number of trials until the first occurrence of consecutive k successes in Bernoulli trials with success probability p which they called a negative binomial distribution of order k . This distribution have caught the interested of researchers, owing to the work of Philippou *et al.*, (1983) whose contribution in this field seems to be very important. Philippou *et al.* (1984) studied a negative binomial distribution of order k and defined a Poisson distribution of order k , attention has been paid to interrelationships among the so called discrete distributions of order k and their exact distribution theory has been developed extensively by many researchers. For any fixed positive integer k , let N_k denote the number of independent trials with success probability p ($q = 1 - p$) until the first occurrence of consecutive k successes. Philippou gave the following definition for the probability mass function of the event $\{N_k = x\}$ (where $\{N_1 = x\}$ reduces to the standard geometric distribution);

Definition 1.1 random variable N_k is said to have a geometric distribution of order k with parameter p if the probability mass function (*pmf*) is given by

$$f_k(x) = P(N_k = x) = \sum_{\substack{x_1, x_2, \dots, x_k \\ x_1 + 2x_2 + \dots + kx_k = x - k}} \binom{x_1 + x_2 + \dots + x_k}{x_1, x_2, \dots, x_k} \times p^x \left(\frac{q}{p}\right)^{x_1 + x_2 + \dots + x_k}; \quad x \geq k \quad (1.1)$$

Where the summation is taken over all non-negative integers $x_1, x_2, \dots, x_k$ satisfying the condition $x_1 + 2x_2 + \dots + kx_k = x - k$.

Ever since then, related concepts has been introduced, reliability theory of the consecutive- k -out-of- n :F systems and studied by Kontoleon (1980), Chiang and Niu (1981) and Derman, Lieberman and Ross (1982). Several related concepts on discrete distributions of order k were studied by Lambiris and Papastavridis (1985), Fu (1985;1986a;1986b), Fu and Beihua (1987), Aki

(1985), Hirano (1986), Philippou (1986), Papastavridis (1987), Aki and Hirano (1988; 1989; 1996; 1997), Chrysaphinou and Papastavridis (1990c), Fu and Koutras (1994a;1994b), Godbole (1990a; 1991) , Griffith (1983), Hirano and Aki (1993), Charalambides (1994) and Makri and Philippou (1994), Feller (1968). Various modifications have been made also on the underlying sequence, e.g. replacing the Bernoulli trials for other random sequences such as some urn models, a binary sequence of order k as in Aki (1985), Aki and Hirano (1994;1995), Dhar and Jiang (1995) and Balakrishnan (1997). Recently, Shao and Fu (2015) derived the same distribution using the transition probability flow graphs (TPFG) method.

It is quite interesting how the classical geometric distribution and related discrete distributions has been generalized and developed by above authors. However, these generalizations of classical geometric distribution is by no mean exhaustive, hence, in this research work, we introduce a new distribution function for a class of arbitrary sequence of S, F (where S and F means success and failure) events that modifies and generalize the geometric distribution of order k . Now, let $E_x(i_m, j_m)$ be a sequence of S, F (binary) events that terminate at the x^{th} trial in which repetition is allowed. If we suppose that this terminal point, is the point at which the event of interest occurs (success). For a given x^{th} terminal point of trials in which the event of interest occurs, observe that this sequence of event forms a partition on x . Suppose $E_x(i_m, j_m)$ partitioned x into k -parts such that $E_x(i_m, j_m)$ repeat itself x_m time ($m = 1, 2, \dots, k+1$) then we consider the sequence of events $\{E_x(i_m, j_m)\}_{x=1}^{\infty}$ with support on M define as follows

$$E_x(i_m, j_m) = (S^{i_m} F^{j_m})^{x_m}; i_m, j_m \geq 0; m \in M, x \geq 1 \quad (1.2)$$

Where i_m, j_m are non-negative integer functions of m , with support on $M = \{1, 2, \dots, k+1\}$.

Definition 1.2 Let $N_{i_{k+1}, j_{k+1}}$ be a random variable that counts the number of trials until the occurrence of event of interest $E_x(i_{k+1}, j_{k+1})$ in (1.2) with success probability p ($q = 1 - p$), then we say that $N_{i_{k+1}, j_{k+1}}$ follows a modified geometric distribution of order (i_{k+1}, j_{k+1}) denoted by $MG_{i_{k+1}, j_{k+1}}(\cdot; p)$ if the probability mass function (pmf) is given by

$$f_{i_{k+1}, j_{k+1}}(x) = P(N_{i_{k+1}, j_{k+1}} = x) = \sum_{\substack{x_1, x_2, \dots, x_k; i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k \ni \\ \sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x - i_{k+1} - j_{k+1}}} \binom{x_1 + x_2 + \dots + x_k}{x_1, x_2, \dots, x_k} \times p^x \left(\frac{q}{p}\right)^{\sum_{m=1}^k j_m x_m + j_{k+1}}; x \geq i_{k+1} + j_{k+1} \quad (1.3)$$

Where the summation is taken over all non-negative integers $x_1, x_2, \dots, x_k; i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k$ satisfying the condition

$$\sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x - i_{k+1} - j_{k+1}; p^{i_{k+1}} q^{j_{k+1}} = 1 - \sum_{m=1}^k p^{i_m} q^{j_m}$$

Then we say that $f_{i_{k+1}, j_{k+1}}$ is a geometric distribution of order (i_{k+1}, j_{k+1}) .

Is easy to see that (1.3) is a proper probability distribution, and moreover, is general than (1.1) . To see this, if we take a particular case of (1.2) and define

$$j_m = \begin{cases} 1, & m = 1, 2, \dots, k \\ 0, & m \geq k+1 \end{cases} \quad \text{and} \quad i_m = m - 1 \quad (1.4)$$

So that (1.2) becomes

$$E(m-1, 1) = S^{m-1} F; m = 1, 2, \dots, k \text{ and } E(k, 0) = S^k \quad (1.5)$$

If we substitute the definitions for i_m and j_m in (1.4) into (1.3) , we easily obtain (1.1) which are results due to Philippou *et al.* (1983).

Theorem 1.3 The probability generating function ($p.g.f.$) of $N_{i_{k+1}, j_{k+1}}$ distributed as $MG_{i_{k+1}, j_{k+1}}(\cdot; p)$ is given by

$$G_{N_{i_{k+1}, j_{k+1}}}(t) = \frac{\pi_{k+1} t^{\delta_{k+1}}}{1 - \sum_{m=1}^k \pi_m t^{\delta_m}}; |t| \leq 1 \quad (1.6)$$

Where $\pi_m = p^{i_m} q^{j_m}$ and $\delta_m = i_m + j_m$

Proof.

$$\begin{aligned} G_{N_{i_{k+1}, j_{k+1}}}(t) &= \sum_{x=i_{k+1}+j_{k+1}}^{\infty} t^x P(N_{i_{k+1}, j_{k+1}} = x) \\ &= \sum_{x=i_{k+1}+j_{k+1}}^{\infty} \sum_{\substack{x_1, x_2, \dots, x_k; i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k \ni \\ \sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x - i_{k+1} - j_{k+1}}} \binom{x_1 + x_2 + \dots + x_k}{x_1, x_2, \dots, x_k} \times p^x \left(\frac{q}{p}\right)^{\sum_{m=1}^k j_m x_m + j_{k+1}} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{x=0}^{\infty} \sum_{\substack{x_1, x_2, \dots, x_k; i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k \ni \\ \sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x}} \binom{x_1 + x_2 + \dots + x_k}{x_1, x_2, \dots, x_k} (pt)^{x+i_{k+1}+j_{k+1}} \left(\frac{q}{p}\right)^{\sum_{m=1}^k j_m x_m + j_{k+1}} \\
 &= (pt)^{i_{k+1}+j_{k+1}} \left(\frac{q}{p}\right)^{j_{k+1}} \sum_{x=0}^{\infty} \sum_{\substack{x_1, x_2, \dots, x_k \\ x_1 + x_2 + \dots + x_k = x}} \binom{x_1 + x_2 + \dots + x_k}{x_1, x_2, \dots, x_k} \times \\
 &\quad (pt)^{\sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m} \left(\frac{q}{p}\right)^{\sum_{m=1}^k j_m x_m} \\
 &= (pt)^{i_{k+1}} (qt)^{j_{k+1}} \sum_{x=0}^{\infty} \sum_{\substack{x_1, x_2, \dots, x_k \\ x_1 + x_2 + \dots + x_k = x}} \binom{x_1 + x_2 + \dots + x_k}{x_1, x_2, \dots, x_k} (pt)^{\sum_{m=1}^k i_m x_m} (qt)^{\sum_{m=1}^k j_m x_m} \\
 &= (pt)^{i_{k+1}} (qt)^{j_{k+1}} \sum_{x=0}^{\infty} \left(\sum_{m=1}^k (pt)^{i_m} (qt)^{j_m} \right)^x \\
 &= \frac{(pt)^{i_{k+1}} (qt)^{j_{k+1}}}{1 - \sum_{m=1}^k (pt)^{i_m} (qt)^{j_m}} = \frac{(pt)^k (1-pt)}{1-t+qt(pt)^k} \quad ; |t| \leq 1 \quad (1.7)
 \end{aligned}$$

Observe that $MG_{i_{k+1}, j_{k+1}}(\cdot; p)$ is a proper probability distribution, which can be easily deduced from (1.7).

Corollary 1.3 Let $G_{N_{i_{k+1}, j_{k+1}}}(t)$ be as in theorem 1.3, if in particular we define

$$j_m = \begin{cases} 1, & m = 1, 2, \dots, k \\ 0, & m \geq k+1 \end{cases} \quad \text{and} \quad i_m = m - 1.$$

Then

$$G_{N_{k,0}}(t) = \frac{(pt)^k (1-pt)}{1-t+qt(pt)^k} \quad (1.8)$$

Equation (1.8) has been obtained by Feller (1968), Philippou *et al.* (1983), Aki (1994), Shao and Wang (2015) via different approach. In the remaining part of this work, the results of our corollaries are result obtained by some of the above mentioned authors.

Theorem 1.4 Let $N_{i_{k+1}, j_{k+1}}$ be as in definition 1.2. Then

$$\begin{aligned}
 \text{(i)} \quad E(N_{i_{k+1}, j_{k+1}}) &= \delta_{k+1} + \frac{\sum_{m=1}^k \delta_m \pi_m}{\pi_{k+1}} \\
 \text{(ii)} \quad Var(N_{i_{k+1}, j_{k+1}}) &= \frac{\sum_{m=1}^k \delta_m (\delta_m - 1) \pi_m}{\pi_{k+1}} + (1 - 3\delta_{k+1}) \frac{\sum_{m=1}^k \delta_m \pi_m}{\pi_{k+1}} - \delta_{k+1}^2 \\
 &\quad - \left(\frac{\sum_{m=1}^k \delta_m \pi_m}{\pi_{k+1}} \right)^2
 \end{aligned}$$

Where $\pi_m = p^{i_m} q^{j_m}$ and $\delta_m = i_m + j_m$

Proof.

$$\text{(i)} \quad \text{It is well known that } E(N_{i_{k+1}, j_{k+1}}) = \frac{\partial G_{N_{i_{k+1}, j_{k+1}}}(t)}{\partial t} \Big|_{t=1}. \text{ But}$$

$$\begin{aligned}
 &\frac{\partial G_{N_{i_{k+1}, j_{k+1}}}(t)}{\partial t} \\
 &= (i_{k+1} + j_{k+1}) p^{i_{k+1}} q^{j_{k+1}} t^{i_{k+1}+j_{k+1}-1} \left(1 - \sum_{m=1}^k p^{i_m} q^{j_m} t^{i_m+j_m} \right)^{-1} + \\
 &\quad p^{i_{k+1}} q^{j_{k+1}} t^{i_{k+1}+j_{k+1}} \left(1 - \sum_{m=1}^k p^{i_m} q^{j_m} t^{i_m+j_m} \right)^{-2} \times \\
 &\quad \left(\sum_{m=1}^k (i_m + j_m) p^{i_m} q^{j_m} t^{i_m+j_m-1} \right) \\
 &= \delta_{k+1} t^{-1} + \frac{\sum_{m=1}^k \delta_m \pi_m t^{\delta_m-1}}{\pi_{k+1} t^{\delta_m}}
 \end{aligned}$$

Hence, we have that

$$\frac{\partial G_{N_{i_{k+1}, j_{k+1}}}(t)}{\partial t} \Big|_{t=1} = \delta_{k+1} + \frac{\sum_{m=1}^k \delta_m \pi_m}{p^{i_{k+1}} q^{j_{k+1}}}$$

(ii) Observe that

$$\frac{\partial^2 G_{N_{i_{k+1}, j_{k+1}}}(t)}{\partial t^2} = -\delta_{k+1} t^{-2} + \frac{\sum_{m=1}^k \delta_m (\delta_m - 1) \pi_m t^{\delta_m - 2}}{\pi_{k+1} t^{\delta_{k+1}}} - \frac{\delta_{k+1} \pi_{k+1} t^{\delta_{k+1} - 1} \sum_{m=1}^k \delta_m \pi_m t^{\delta_m - 2}}{(\pi_{k+1} t^{\delta_{k+1}})^2}$$

Thus,

$$\frac{\partial^2 G_{N_{i_{k+1}, j_{k+1}}}(t)}{\partial t^2} \Big|_{t=1} = -\delta_{k+1} + \frac{\sum_{m=1}^k \delta_m (\delta_m - 1) \pi_m}{\pi_{k+1}} - \frac{\delta_{k+1} \sum_{m=1}^k \delta_m \pi_m}{\pi_{k+1}}$$

Using the result in (i) above and after some algebraic simplification, it follows that

$$\begin{aligned} \text{Var}(N_{i_{k+1}, j_{k+1}}) &= \frac{\partial^2 G_{N_{i_{k+1}, j_{k+1}}}(t)}{\partial t^2} \Big|_{t=1} + \frac{\partial G_{N_{i_{k+1}, j_{k+1}}}(t)}{\partial t} \Big|_{t=1} - \left(\frac{\partial G_{N_{i_{k+1}, j_{k+1}}}(t)}{\partial t} \Big|_{t=1} \right)^2 \\ &= \frac{\sum_{m=1}^k \delta_m (\delta_m - 1) \pi_m}{\pi_{k+1}} + (1 - 3\delta_{k+1}) \frac{\sum_{m=1}^k \delta_m \pi_m}{\pi_{k+1}} - \delta_{k+1}^2 - \left(\frac{\sum_{m=1}^k \delta_m \pi_m}{\pi_{k+1}} \right)^2 \end{aligned}$$

Observe that by theorem 1.4, $E(N_{k,0}) = k + \frac{q}{p^k} \left[\frac{1-p^k-kp^k+kp^{k+1}}{(1-p)^2} \right]$ is a special case, hence we state the following corollary.

Corollary 1.5 Let $N_{i_{k+1}, j_{k+1}}$ be as in definition 1.2. If in particular we define

$$j_m = \begin{cases} 1, & m = 1, 2, \dots, k \\ 0, & m \geq k+1 \end{cases} \quad \text{and} \quad i_m = m - 1.$$

Then

$$\begin{aligned} \text{(i)} \quad E(N_{k,0}) &= \frac{[1-p^k]}{qp^k} \\ \text{(ii)} \quad \text{Var}(N_{k,0}) &= \frac{[1-(2k+1)qp^k-p^{2k+1}]}{q^2 p^k} \end{aligned}$$

Definition 1.6 Let $N^1_{i_{k+1}, j_{k+1}}, N^2_{i_{k+1}, j_{k+1}}, \dots, N^r_{i_{k+1}, j_{k+1}}$ be independent identically distributed random variables. Then the distribution of $N_{i_{k+1}, j_{k+1}, r} = N^1_{i_{k+1}, j_{k+1}} + N^2_{i_{k+1}, j_{k+1}} + \dots + N^r_{i_{k+1}, j_{k+1}}$ is called a modified negative binomial distribution of order (i_{k+1}, j_{k+1}) denoted by $MNB_{i_{k+1}, j_{k+1}}(\cdot; p)$

The following theorem is a consequence of definition 1.6.

Theorem 1.7 Let $N^1_{i_{k+1}, j_{k+1}}, N^2_{i_{k+1}, j_{k+1}}, \dots, N^r_{i_{k+1}, j_{k+1}}$ be independent identically distributed random variables. Then the probability generating function of $N_{i_{k+1}, j_{k+1}, r} = N^1_{i_{k+1}, j_{k+1}} + N^2_{i_{k+1}, j_{k+1}} + \dots + N^r_{i_{k+1}, j_{k+1}}$

$$G_{N_{i_{k+1}, j_{k+1}, r}}(t) = \left(\frac{\pi_{k+1} t^{\delta_{k+1}}}{1 - \sum_{m=1}^k \pi_m t^{\delta_m}} \right)^r; |t| \leq 1, \quad (1.9)$$

Where $\pi_m = p^i q^j$, $\delta_m = i_m + j_m$.

Proof.

Since $N^1_{i_{k+1}, j_{k+1}}, N^2_{i_{k+1}, j_{k+1}}, \dots, N^r_{i_{k+1}, j_{k+1}}$ are independent identically distributed random variables with $p.g.f.$ $G_{N^j_{i_{k+1}, j_{k+1}}}$ (as in (1.6)) for each $j \in \{1, 2, \dots, r\}$, thus it follows that

$$\begin{aligned} G_{N_{i_{k+1}, j_{k+1}, r}}(t) &= \prod_{j=1}^r G_{N^j_{i_{k+1}, j_{k+1}}}(t) \\ &= \left(G_{N^j_{i_{k+1}, j_{k+1}}}(t) \right)^r = \left(\frac{\pi_{k+1} t^{\delta_{k+1}}}{1 - \sum_{m=1}^k \pi_m t^{\delta_m}} \right)^r \end{aligned}$$

Theorem 1.8 Let $N_{i_{k+1}, j_{k+1}, r}$ be as in definition 1.6, then the probability $\{N_{i_{k+1}, j_{k+1}, r} = x\}$ is given by

$$\begin{aligned} f_{i_{k+1}, j_{k+1}, r}(x) &= P(N_{i_{k+1}, j_{k+1}, r} = x) = \\ &\sum_{\substack{x_1, x_2, \dots, x_k; i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k \ni \\ \sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x - (i_{k+1} - j_{k+1})r}} \binom{x_1 + x_2 + \dots + x_k + r - 1}{x_1, x_2, \dots, x_k, +r-1} \times \\ &p^x \left(\frac{q}{p} \right)^{\sum_{m=1}^k j_m x_m + r j_{k+1}}; x \geq (i_{k+1} + j_{k+1})r \quad (1.10) \end{aligned}$$

Where the summation is taken over all non-negative integers $x_1, x_2, \dots, x_k; i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k$ such that

$$\sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x - (i_{k+1} - j_{k+1})r$$

Proof.

By theorem 1.7 the *p.g.f.* of $N_{i_{k+1}, j_{k+1}, r}$ is given by

$$G_{N_{i_{k+1}, j_{k+1}, r}}(t) = \left(\frac{\pi_{k+1} t^{\delta_{k+1}}}{1 - \sum_{m=1}^k \pi_m t^{\delta_m}} \right)^r ; |t| \leq 1, \pi_m = p^{i_m} q^{j_m}, \delta_m = i_m + j_m$$

Hence by binomial expansion we have that

$$\begin{aligned} G_{N_{i_{k+1}, j_{k+1}, r}}(t) &= (\pi_{k+1} t^{\delta_{k+1}})^r \sum_{x=0}^{\infty} \binom{-r}{x} \left(- \sum_{m=1}^k \pi_m t^{\delta_m} \right)^x \\ &= (\pi_{k+1} t^{\delta_{k+1}})^r \sum_{x=0}^{\infty} \binom{x+r-1}{r-1} \sum_{\substack{x_1, x_2, \dots, x_k \geq 0 \\ x_1 + x_2 + \dots + x_k = x}} \binom{x}{x_1, x_2, \dots, x_k} \prod_{m=1}^k (\pi_m t^{\delta_m})^{x_m} \\ &= (p^{i_{k+1}} q^{j_{k+1}} t^{\delta_{k+1}})^r \sum_{x=0}^{\infty} \binom{x+r-1}{r-1} \sum_{\substack{x_1, x_2, \dots, x_k \geq 0 \\ x_1 + x_2 + \dots + x_k = x}} \binom{x}{x_1, x_2, \dots, x_k} \prod_{m=1}^k (p^{i_m} q^{j_m} t^{\delta_m})^{x_m} \\ &= (p^{i_{k+1}} q^{j_{k+1}} t^{\delta_{k+1}})^r \sum_{x=0}^{\infty} \sum_{\substack{x_1, x_2, \dots, x_k; i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k \geq 0 \\ \sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x}} \binom{x+r-1}{r-1} \binom{x}{x_1, x_2, \dots, x_k} \times \\ &\quad (pt)^{\sum_{m=1}^k i_m x_m} (qt)^{\sum_{m=1}^k j_m x_m} \\ &= (pt)^{ri_{k+1} + rj_{k+1}} \left(\frac{q}{p} \right)^{rj_{k+1}} \sum_{x=0}^{\infty} \sum_{\substack{x_1, x_2, \dots, x_k \geq 0 \\ \sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x}} \binom{x_1 + x_2 + \dots + x_k + r - 1}{x_1, x_2, \dots, x_k, +r-1} \\ &\quad \times (pt)^{\sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m} \left(\frac{q}{p} \right)^{\sum_{m=1}^k j_m x_m} \\ &= \sum_{x=ri_{k+1} + rj_{k+1}}^{\infty} \sum_{\substack{x_1, x_2, \dots, x_k; i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k \geq 0 \\ \sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x - ri_{k+1} + rj_{k+1}}} \binom{x_1 + x_2 + \dots + x_k + r - 1}{x_1, x_2, \dots, x_k, +r-1} \times \\ &\quad (pt)^x \left(\frac{q}{p} \right)^{\sum_{m=1}^k j_m x_m + rj_{k+1}} \\ &= \sum_{x=ri_{k+1} + rj_{k+1}}^{\infty} t^x P(N_{i_{k+1}, j_{k+1}, r} = x) \end{aligned}$$

Hence the result follows immediately.

Corollary 1.9 Let $N_{i_{k+1}, j_{k+1}, r}$ be as in definition 1.6. If in particular we define

$$j_m = \begin{cases} 1, & m = 1, 2, \dots, k \\ 0, & m \geq k+1 \end{cases} \quad \text{and} \quad i_m = m - 1. \quad \text{Then}$$

$$\begin{aligned} P(N_{k,0,r} = x) &= \sum_{\substack{x_1, x_2, \dots, x_k \geq 0 \\ \sum_{m=1}^k (m-1)x_m + \sum_{m=1}^k x_m = x - rk}} \binom{x_1 + x_2 + \dots + x_k + r - 1}{x_1, x_2, \dots, x_k, r-1} \times \\ &\quad p^x \left(\frac{q}{p} \right)^{\sum_{m=1}^k x_m} ; x \geq kr \quad (1.11) \end{aligned}$$

Where the summation is taken over all non-negative integers $x_1, x_2, \dots, x_k$; satisfying the condition $\sum_{m=1}^k (m-1)x_m + \sum_{m=1}^k x_m = x - kr$.

Definition 1.10. Let $N_{i_{k+1}, j_{k+1}, \mu}$ be a discrete random variables and μ a nonnegative integer constant. Assuming that $q \rightarrow 0$ and $r q \rightarrow \mu$ as $r \rightarrow \infty$. Then $N_{i_{k+1}, j_{k+1}, r} \rightarrow N_{i_{k+1}, j_{k+1}, \mu}$. We say that $N_{i_{k+1}, j_{k+1}, \mu}$ has the modified Poisson distribution of order (i_{k+1}, j_{k+1}) denoted by $MP_{i_{k+1}, j_{k+1}}(\cdot; \mu)$.

Theorem 1.11 Let $N_{i_{k+1}, j_{k+1}, \mu}$ be as in definition 1.10, then the probability $\{N_{i_{k+1}, j_{k+1}, \mu} = x\}$ is given by

$$\begin{aligned} \lim_{r \rightarrow \infty} P(N_{i_{k+1}, j_{k+1}, r} - ri_{k+1} + rj_{k+1} = x) &\rightarrow P(N_{i_{k+1}, j_{k+1}, \mu} = x) = \\ &\sum_{\substack{x_1, x_2, \dots, x_k; i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k \geq 0 \\ \sum_{m=1}^k i_m x_m + \sum_{m=1}^k j_m x_m = x}} e^{-\mu i_{k+1}} \frac{\mu^{\sum_{m=1}^k j_m x_m}}{x_1! x_2! \dots x_k!} \quad \forall i_m \geq 0 \quad (1.12) \end{aligned}$$

Proof.

It suffices to show that the characteristic function of $N_{i_{k+1}, j_{k+1}, r} - ri_{k+1} + rj_{k+1}$ converges to $MP_{i_{k+1}, j_{k+1}}(\cdot; \mu)$ as $r \rightarrow \infty$. From (1.9) we have

$$\phi_{N_{i_{k+1}, j_{k+1}, r}}(t) = \left(\frac{\pi_{k+1} \exp(\delta_{k+1} it)}{1 - \sum_{m=1}^k \pi_m \exp(\delta_m it)} \right)^r$$

Since $|q^{j_{k+1}} \exp(\delta_{k+1} it)| \leq 1$ for $|t| \leq 1$. It then follows that

$$\begin{aligned} \phi_{N_{i_{k+1}, j_{k+1}, r}}(t) &\leq \left(\frac{p^{i_{k+1}}}{1 - \sum_{m=1}^k \pi_m \exp(\delta_m it)} \right)^r \\ &= \frac{\left(1 - \frac{\mu}{r}\right)^{ri_{k+1}}}{(1 - r^{-j_m} \sum_{m=1}^k p^{i_m} \mu^{j_m} \exp(\delta_m it))^r} \end{aligned}$$

By hypothesis, we have that $q \rightarrow 0$ and $rq \rightarrow \mu$ as $r \rightarrow \infty$. Thus, as $r \rightarrow \infty$ we have that

$$\begin{aligned} \lim_{r \rightarrow \infty} \phi_{N_{i_{k+1}, j_{k+1}, r}}(t) &\leq \lim_{r \rightarrow \infty} \frac{\left(1 - \frac{\mu}{r}\right)^{ri_{k+1}}}{(1 - r^{-j_m} \sum_{m=1}^k p^{i_m} \mu^{j_m} \exp(\delta_m it))^r} \\ &= \exp\left(-\mu i_{k+1} + \sum_{m=1}^k \mu^{j_m} \exp(\delta_m it)\right) \end{aligned}$$

Hence the result follows immediately.

Corollary 1.9 Let $N_{i_{k+1}, j_{k+1}, \mu}$ be as in definition 1.10. If in particular we define

$$j_m = \begin{cases} 1, & m = 1, 2, \dots, k \\ 0, & m \geq k+1 \end{cases} \quad \text{and} \quad i_m = m - 1.$$

Then

$$P(N_{i_{k+1}, 0, \mu} = x) = \sum_{\substack{x_1, x_2, \dots, x_k \\ \sum_{m=1}^k (m-1)x_m + \sum_{m=1}^k x_m = x}} e^{-\mu k} \frac{\mu^{\sum_{m=1}^k x_m}}{x_1! x_2! \dots x_k!}$$

Conclusion

We presented a new modified distribution function whose order of sequence of runs of success-failure is an arbitrary function of k , this modifies, generalize and complement the usual geometric distribution of order k studied by several authors in the literature. The associated statistics, generating function and its relationship with some other discrete distributions using standard augment and convergence analysis are established.

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